Braidings from Braids and Weavings

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Abstract

We investigate how to replace the threads in a weaving by braided threads and how to interweave them with each other. Based on our findings, we introduce a subdivision scheme applicable to interwoven braids made of an arbitrary number of threads, leading to a fractal-like output. We realize our results both digitally and in handmade form.

Braidings and Weavings

Braiding and weaving are two techniques used to create resistant materials like ropes, ribbons, and fabric from threads, or baskets and chair seats from plant-based materials. Relics dated back to the New Stone Age (Neolithic) show mankind is familiar with weaving longer than with pottery. Both techniques allow for a broad variety of patterns by changing small steps in the manufacturing process. Especially the visual appearance changes drastically. This can be seen by comparing textile fabric made in plain weave and in $2-2$ twill, as shown in Figure 1. The plain weave is a simple weaving pattern, where the thread in weft direction goes above and below the threads in warp direction alternatingly, while the next row is worked in reversed order. Varying the number of consecutive warp threads that a weft thread passes above or below leads to patterns like $2-2$ twill. This leads to a more intricate structure showing stripes or zigzag lines. When a weaving is done with same-colored threads in both warp and weft direction, the weave pattern determines its tactile and visual structure. Patterns can be introduced by using different colors for the threads. For instance, separate colors can be used for warp and weft direction as shown in Figure 1. A next step would be to vary colors within warp and/or weft direction, which yields stripes, checked patterns, or pied–de–poule.

Braids are made from an arbitrary number of threads, at least three, from flexible material like hair or yarn. Braidings can be made flat by turning the threads at the boundaries back into the braid, creating a ribbon-like structure. Alternatively, threads can continue around a cylinder, closing the braid to a cylindrical structure as shown in Figure 1. In the following, we only consider flat, ribbon-like braids. Note that a small part of the final product does not necessarily give away whether it was woven or braided. This can be observed during the process, where the main difference lies in how threads are being interwoven with each other.

Figure 1: *Left to right: plain weave;* $2-2$ *twill; braids with three threads, left made outside-in, right made inside-out; braid with four threads, made outside-in; flat braid; cylindrical braid as rope housing.*

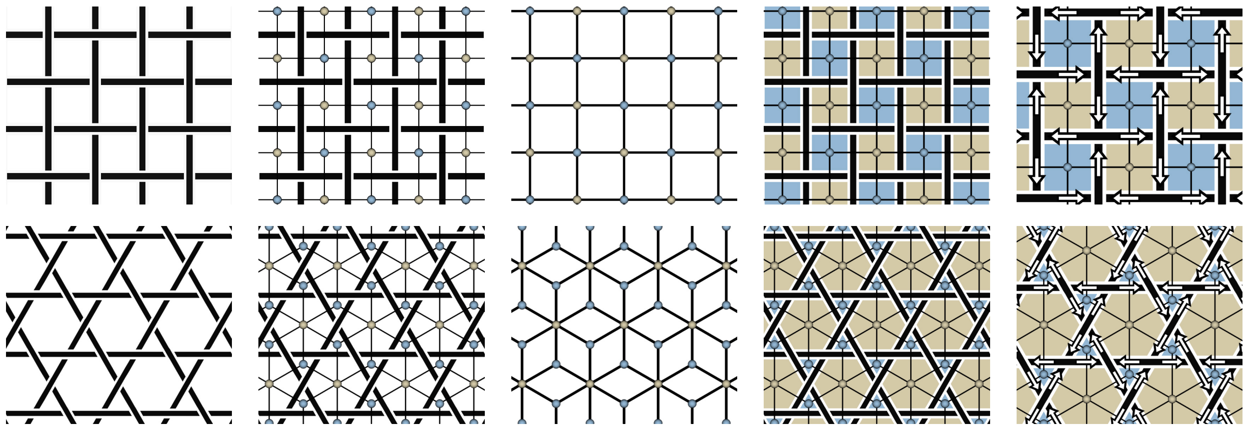

Figure 2: *Upper row: plain weave, lower row: kagome pattern. Both rows, from left to right: pattern; pattern with its dual; dual pattern with 2-colored vertices; 2-coloring of original pattern based on vertices of dual pattern; 2-coloring of original pattern with orientations.*

Choosing a stiffer material, like wood splints, allows for translating weaving and braiding to the production of baskets or carpet-beaters. Here, a popular example is the *kagome* pattern (from Japanese 籠 目, literally: basket/cage—eye). It consists of those threads forming a pattern consisting of regular hexagons and triangles, see Figure 2, lower row. The threads obey the rule to go above and below other threads alternatingly.

Weavings and braidings carry fascinating mathematical properties that have been investigated in several prior works. These focus on specific traditions like woven Colombian hats [3] or the application of kagome patterns to approximate freeform surfaces [1]. Furthermore, the plain weave has been transferred to tilings, such as the square or trihexagonal tiling and to three-dimensional solids, like the Archimedean solids [10]. Further, the pied–de–poule pattern, a twill based on two-colored warp and weft directions, has been combined with a subdivision scheme to create fractal structures [5]. So-called mad weaves are twill weaves based on the kagome pattern [7]. A detailed introduction to geometric aspects of fabrics is given in [8].

Interweaving Braids

In this paper, we investigate how to replace threads of a weaving by a braided set of threads. For that, we assume the underlying weavings to be made of straight line arrangements where no more than two lines meet at the same point. Then, each thread in the replacement braiding should maintain the property of alternately going above and below all threads it meets, independent of whether these are threads from the same braided set or another braided set. Furthermore, we aim for an arbitrary number of threads.

We call all locations in the original line arrangement where a line went above or below another line a *crossing*. Along the segments between the crossings, the replacing set of threads should form a braid. We assume that the line segments between any two crossings are of the same length. This ensures that we can produce a faithful representation, which would not be possible if the lengths scaled by some irrational factor.

If we also assume symmetric angles, this reduces our choice of original line arrangements to those that make up certain Archimedean tilings. From these, we will focus on the square and trihexagonal tiling, although our observations hold for less regular tilings. In case of the square tiling, the lines are arranged as plain weave, while in case of the trihexagonal tiling, the lines follow the kagome pattern, see Figure 2. For now, we will focus on sets of three braided threads and investigate how to obtain interweavings at crossings.

As shown in Figure 1, a braid can be made in two ways: outside-in or inside-out. Usually, for three threads, we start with taking one of the outer threads, say the right one, and put it above the inner one, now

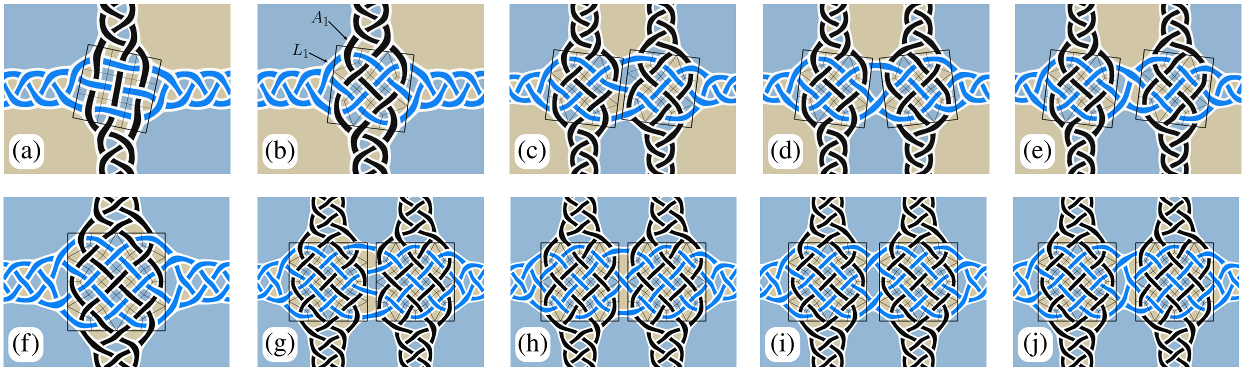

Figure 3: *(a) Two braids crossing orthogonally with a minimal number of nine crossings in blue and black; (b) two braids crossing orthogonally with additional crossings of the same color (nine crossings in blue and black, two in black, two in blue); (c)–(e) two black braids crossing one blue braid with no, one, and two blue crossing(s) in between, following the classical braiding pattern for the blue braid. (f) Two braids crossing orthogonally; (g)–(j) a blue braid crossing two black ones, with no, one, two, and three additional crossing(s).*

taking the middle position. The next step consists of placing the outer thread on the left above the new thread in the middle. Repeating these two steps adds more length to the braid. Alternatively, for the symmetric construction, we can make a braid also by placing the outer threads below the one in the middle. The resulting braids are similar to each other, since they differ solely by a reflection.

Let us consider a single crossing where a braided set L of threads L_1, L_2, L_3, coming from the left, meets a braided set A of threads A_1, A_2, A_3 coming from above, as illustrated in Figure 3 (b). Here, the braid A is colored in black, while L is shown in blue. Two natural ways to follow through the crossing are the plain weave or the braiding pattern. We will first focus on the plain weave. Here, each thread from the braid should run through the interweaving in a straight manner. Entering the crossing, the topmost thread L_1 in L has to interact with the leftmost thread A_1 in A. If the last interaction of A_1 within its braid was to go below another thread, it now has to go above L_1, since we want to maintain the property that they go alternatingly above and below threads they meet. Or, if A_1 just went above another thread in its braid, it now has to go below L_1. Similarly, the last interaction of L_1 demands whether it goes above or below A_1. Out of the four possibilities, only two are *compatible*. Choosing one fixes the entire crossing, as shown in Figure 3 (a).

As an alternative to the plain weave, we can also continue the braiding through the crossing, as shown in Figure 3 (b). Again, once the above/below choice between an interacting blue and black thread has been made, all other interactions fall in place. To us, this forms the more interesting patterns, which is why we choose it as the base-interweaving for all upcoming considerations. Figure 3 is not only applicable to the square tiling, but can be sheared for the trihexagonal tiling. In either case, one braid is made outside-in while the other one is made inside-out. This behavior changes into the opposite after each crossing. Therefore, when choosing how the braids replace the line segments between the crossings, we have to ensure this compatibility, which begs the question: Can we find replacements that allow for all crossings to be interwoven compatibly?

Both tilings are 2-colorable, since the square tiling can be colored in a checkerboard pattern and the trihexagonal tiling can be 2-colored by giving triangles one and hexagons another color, as shown in Figure 2. More generally, this 2-coloring can be found since in both cases, the edge graphs of the tilings are Eulerian and planar. The 2-coloring then induces an orientation on all edges of the tiling. Without loss of generality, we can assume that blue faces are oriented counter-clockwise and beige faces clockwise as shown in Figure 2. This ensures that when choosing the orientation of a single edge, all other edges can be oriented in a fashion that renders all crossings compatible, including the flip in orientation after each crossing when following a

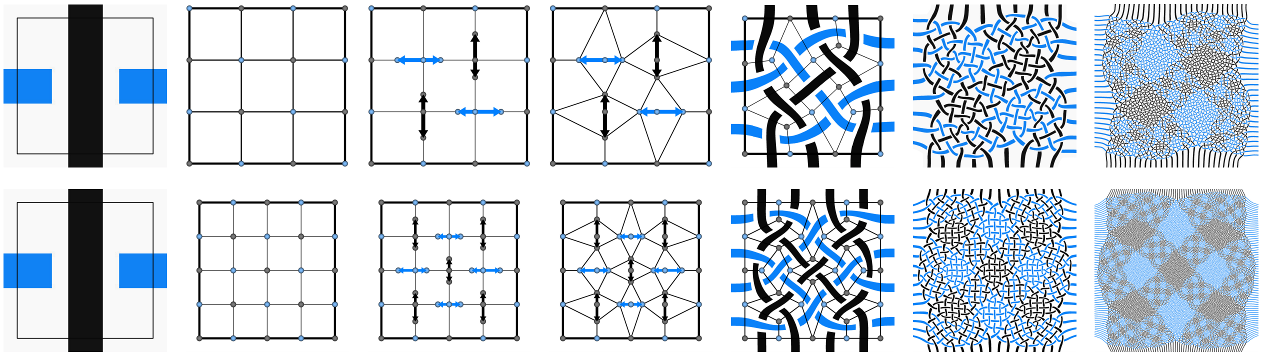

Figure 4: *Subdivision scheme on plain weave on braids with three (upper row) and four (lower row) threads. Both rows, from left to right: crossing of black and blue thread to be subdivided; introduction of auxiliary grid; introduction of edges to be split; introduction of new quadrilaterals by edge split; result of one, two, and three subdivision step(s).*

straight line. Here, braids made along line segments in arrow direction are made inside-out, while braids made along line segments in opposite arrow direction are made outside-in. In terms of the effect of braiding steps between the crossings, adding an odd number of braiding steps between two crossings results in them being reflections of each other, whereas an even number keeps them the same, as shown in Figure 3 (c)–(e).

Generalization and Remarks

As mentioned earlier, the construction described in the previous section can be generalized to braids made of an arbitrary number of threads. Let $n \in \mathbb{N}_{\geq 2}$ denote the number of threads per braid. For odd n, the way crossings are made can be transferred directly from braids made of three threads. Coloring the areas enclosed by the braids, as in Figure 3 (a)–(e) for braids made of an odd number of threads, yields an appropriate 2-coloring of the plane. Here, Figure 3 (b) is rotationally symmetric by the angle $\frac{\pi}{2}$, exchanging the colors of the areas and braids. This observation remains true for any odd number of threads.

Now, let us turn to the case where n is even, and consider $n = 4$. Then, a plain interweaving through the crossing is easily possible, but a braided interweaving yields far more interesting results, as illustrated in Figure 3 (f). The plain crossing created with four threads per line is known in bobbin lace, see [6, p. 25].

Here, we see that the crossing is similar in structure to the one made by braids of three threads each. In contrast to what we observed before, for four (and every other even n) threads per braid, the crossing does not have the same symmetry as in the odd case. Instead of $\frac{\pi}{2}$-rotational symmetry, here, we can apply a mirror reflection to the center of the crossing, which can be seen in the in- and outgoing braids as well. Coloring of the areas enclosed by the braids does not yield the same result as for odd n. To continue the braiding through further crossings, we need to vary the number of crossings in the braids before entering a crossing or after leaving it. Four possible variations are illustrated in Figure 3 (g)–(j).

Readers familiar with bobbin lace may recognize the similarities to our findings. While patterns in bobbin lace are usually designed for an even number of threads, as the bobbins come in pairs, we can generalize our approach to an arbitrary number of threads. Additionally, the braids of three threads can be replaced by two threads, even though two threads do not make a proper braid. This idea has been illustrated previously by deriving weavings from a subdivision scheme applied to arbitrary planar tilings [9]. Hence, using two threads only, the results coincide, even though they are derived in different ways. In case of two threads, similar findings were already presented to the Bridges community [10]. The crossings of braids in Figure 3 are visually similar to celtic knots [2] by connecting the outgoing threads in an appropriate way.

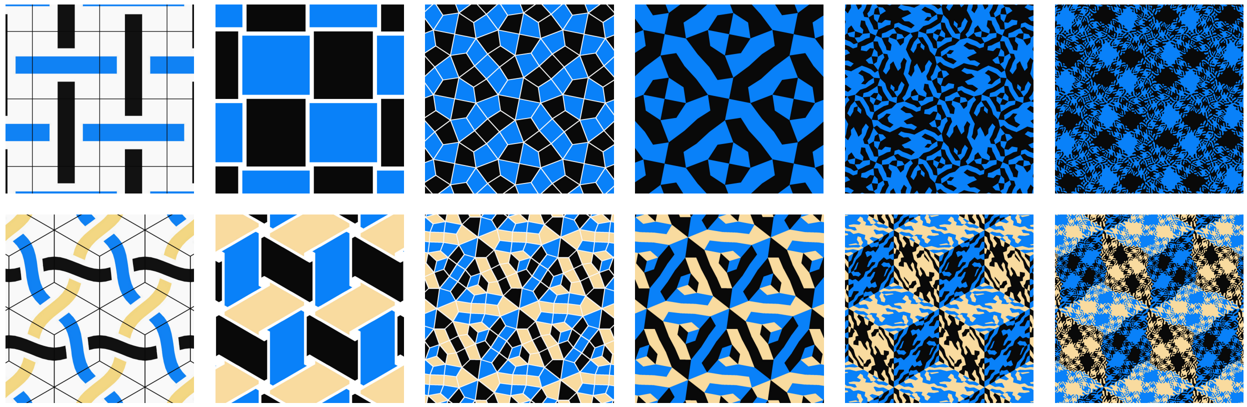

Figure 5: *Application of subdivision scheme to plain weave and to kagome pattern resulting in fractal-like structures. Upper row: plain weave without subdivision, first subdivision step with quadrilaterals shown in white, followed by one, two, and three step(s) of the subdivision scheme applied. Lower row: kagome pattern without subdivision, first subdivision step with quadrilaterals shown in white, followed by one, two, and three step(s) of the subdivision scheme applied.*

Braidings from Subdivision

In this section, we introduce a subdivision scheme applied to the crossings of a braiding as described earlier. We keep the warp direction colored black and the weft direction colored blue and reduce our considerations to the square tiling and the trihexagonal tiling. Figure 2 has the dual tilings of both, which, in both cases, consist of quadrilaterals only. Further, every quadrilateral of the dual tilings contains a crossing of the original weaving or braiding. Up to rotation, four different types of crossings occur. Two consist of threads both in the same color, while the other two have different colors, where blue is above black and the other way round.

The subdivision scheme works as follows: Assume each line will be replaced by a braid made of three threads. After applying the scheme once, there should be two sets of three threads entering the quadrilateral, each via one of its edges and leaving it via the opposite edge. For that, each quadrilateral is replaced by nine smaller ones forming a 3×3-section of a quadrilateral tiling. The twelve edges of the quadrilaterals newly introduced, which do not lie on the edges of the subdivided tile, are called *inner edges*. Accordingly, the four vertices incident to inner edges solely are called *inner vertices*, compare to Figure 4, second left.

To recreate the result of Figure 3 via the subdivision, each inner vertex is replaced by a straight line segment as shown in Figure 4, third left. These line segments do not cross each other and imitate the plain weave pattern in blue and black. The placement of the line segments does not influence the subdivision scheme itself, solely the visual output. From the perspective of a single quadrilateral, this leaves a symmetric choice: the line segments could be placed as in Figure 4 or all vertical black segments could be exchanged by horizontal blue ones and vice versa. While this seems to be a choice, this will actually be fixed once neighboring elements come into play.

In the next step, we connect the end points of the colored line segments indicated by the arrows in Figure 4 to the closest vertices on the edges of the original tile or end points of other line segments, lying on those inner edges orthogonal to the colored line segment. The colored line segments now serve as diagonals of the quadrilaterals newly introduced. These diagonals are then removed, yielding an arrangement of 13 quadrilaterals subdividing the original square. Now, each quadrilateral is filled with a crossing of two threads. A thread always runs from one edge of the quadrilateral to the opposite edge. Some of the new crossings are of mixed colors, while others appear solely in black or blue. Those quadrilaterals having a colored line segment as a diagonal contain crossings in the color of the line segment. The square in the middle shares all edges with quadrilaterals containing crossings each in a single color. Again, one first crossing determines the above/below behavior after which all subsequent interactions fall into place, see Figure 4, third from right.

Since one application of the subdivision scheme yields 13 crossings, we can apply the same scheme recursively. However, each application now has to respect its neighbors. This compatibility is ensured by zero braid steps in between the crossings, shown in Figure 3 (c) for odd, and Figure 3 (g) for even numbers of threads. The even case is shown in the second row of Figure 4. For iterated application, some of the input crossings are of the same color creating larger regions of crossings in a single color. The visual output is reminiscent of previous findings [5]. In case of the kagome pattern, the approach is the same. Here, we do not subdivide a square, but a rhombus, containing the crossing to be subdivided. Instead of squares, the rhombus is now subdivided into 3×3 smaller rhombi, which can be seen as a dilated version of the squares.

In the subdivision step, we can choose any $n \in \mathbb{N}_{\geq 2}$ as the number of threads. Then, in each step of the subdivision scheme, each thread is replaced by n new ones. Hence, after the first step of the subdivision, $2n$ threads meet at each crossing. For a single crossing, we first introduce n^2 (dilated) squares and replace via the split of the colored line segments each of the $(n-1)^2$ inner vertices by a quadrilateral. By construction, each square is maintained as a(n irregular) quadrilateral. In total, there are $2n(n-1)+1$ many quadrilaterals, and hence crossings. The quadrilateral in the middle arises either through the edge splits around the middle square if n is odd, or by replacing an inner vertex by a quadrilateral, if n is even. In the case crossings of threads colored differently, there are n^2 crossings in mixed colors and $\left\lceil \frac{1}{2}(n-1)^2 \right\rceil$ in the color on top of the

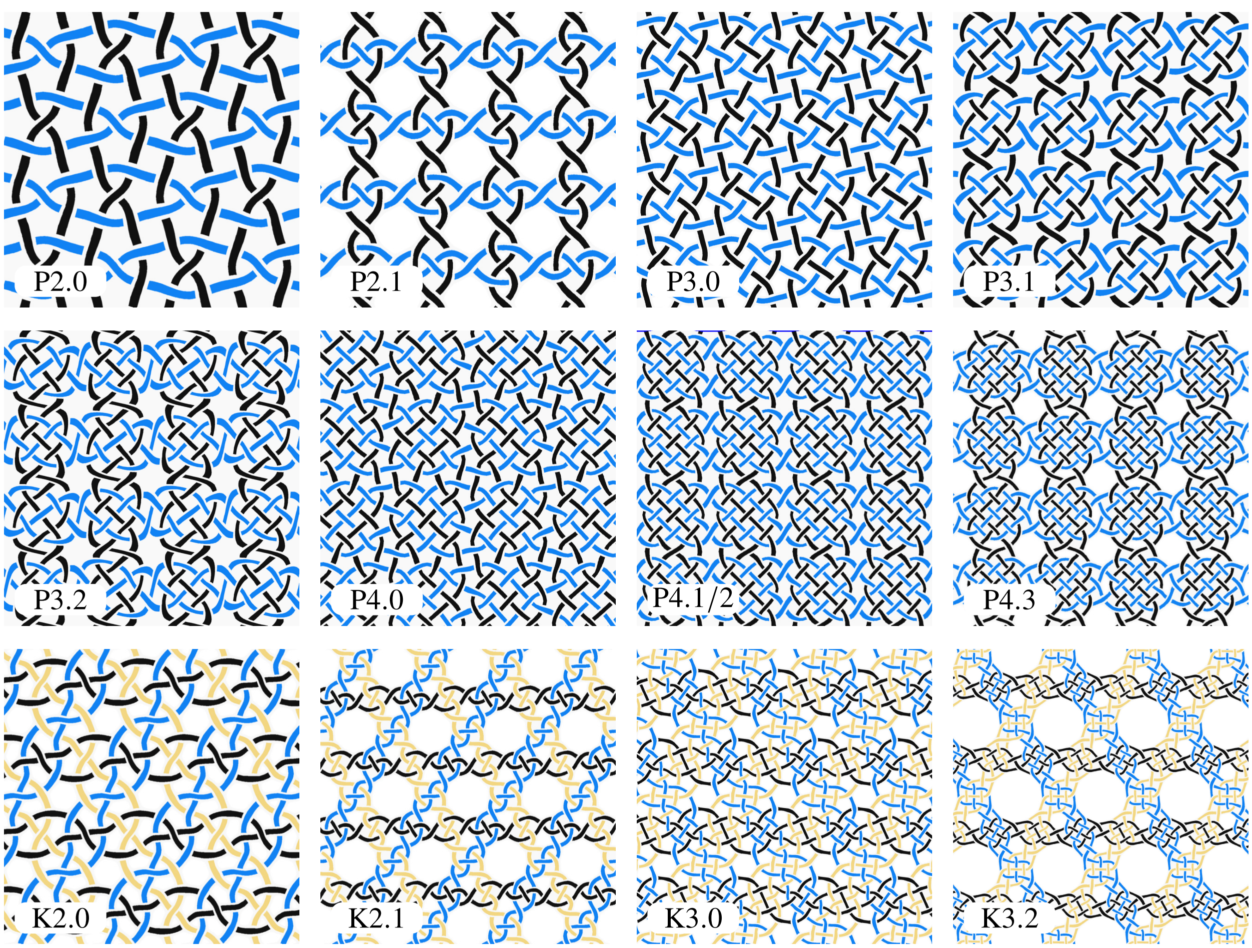

Figure 6: *Examples of interwoven braids with various numbers of threads on plain weave and kagome.*

crossing subdivided, while the remaining crossings are colored in the color of the thread below.

Figure 5 shows three steps of the subdivision scheme applied to both tilings, with the directions colored in blue, black, and sand, respectively. In contrast to the illustrations shown so far, which depict the threads used by small gaps, letting the underground shine through, here, we show a compressed representation consisting of quadrilaterals only. This is inspired by the simplest representation of the plain weave, which is made of squares. Hence, every quadrilateral is filled with the color of the thread on top of the crossing contained in it. Because the coloring of a single crossing determines the coloring of the subdivided crossing, we derive the subdivided one from the representation shown in Figure 4 and color the quadrilaterals afterwards accordingly.

Results

Figure 6 shows a collection of examples made of two to four threads per braid. To list the properties of each presented result, we introduce the following naming scheme: The capital letter denotes the underpinning weaving—P denotes plain weave, while K denotes kagome. It is followed by the number of threads per braid. Separated by a period, we list how many crossings following the classical braiding pattern are made between two crossings of braids. If there are different numbers of classical braiding crossings in warp and weft direction, they are separated by a slash. Thus, a braiding on plain weave made of three threads with no additional crossing is denoted by P3.0. One may wonder why there is no K3.1 example in Figure 6. This

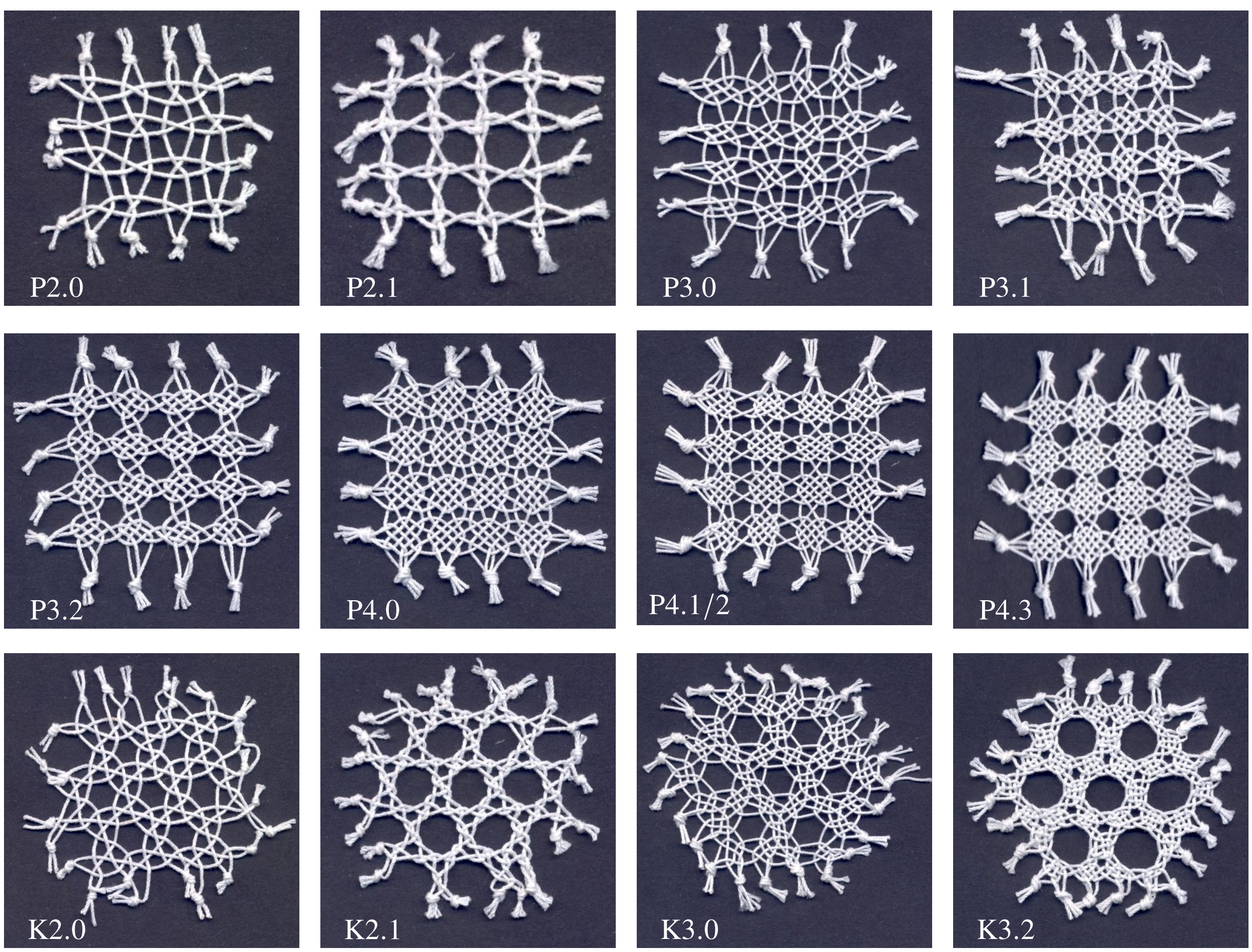

Figure 7: *Handmade results corresponding to those made digitally, shown in Figure 6.*

is because adding only one braiding step between two crossings results in two crossings being reflections of each other, as shown in Figure 3 (d), which is incompatible with the threefold symmetry at each crossing.

In Figure 7, we show handmade realizations of the examples depicted in Figure 6, which are all made from white cotton yarn. The examples shown in Figure 7 are created in a single color in contrast to the design drawings in Figure 6 to emphasize the overall structure rather than the different sets of parallel braids.

It is possible to craft a crossing area like in Figure 3 (b) or 3 (f), defined by the subdivision scheme, solely by crossing threads—it is never necessary to lead a thread through the already braided area. So, if the input to the subdivision already has the property that it can be generated in an appropriate order by crossings only, this property is also kept for the braiding defined by an additional subdivision step. Also braid parts between crossing areas like in Figure 3 (c)–(e) or 3 (g)–(j) do not destroy this property. Since all the examples are made based on the plain weave and kagome pattern, all of them can be done solely by crossing threads. During the manufacturing process, the physical realizations were pinned on a cork panel and fixed with pins to keep the crossings in place. An advanced manufacturer may create a denser result without using pins.

Conclusions and Future Work

We investigated how threads in a weaving could be replaced by braids and presented several ways to interweave braids with each other, creating plane-filling patterns. Three aspects remain as future work. First, handmade results can be created for the subdivision parts as well as for other tilings that satisfy our assumptions. Second, it would be desirable to extend the construction to tilings like the regular triangle grid, where three lines meet in a point. Third and finally, some tilings like the 6–4–3–4 tiling induce threads closing in on themselves. This creates a connectivity akin to chain mail [4], a connection yet to be explored.

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