Illustrating Mathematics

Why and how to get involved

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Imagine transforming abstract mathematical concepts into captivating visual forms: fractals that spiral infinitely, geometric shapes that reveal hidden symmetries, or visual proofs that make complex ideas instantly intuitive. This is the world of mathematical illustration, where equations and theorems meet art and design, making the invisible visible. In this article, we'll explore how researchers and enthu- siasts use creative visuals to communicate math's beauty and how mathematical illustrations are currently growing to become a field in its own right. Finally, we will share how you can make the first steps to begin illustrating mathematics yourself.

We will start with an example. A fundamental concept in calculus or analysis is series. Here, the object of consideration is an infinite sum of values, for instance:

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots=\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$$

Two typical questions are: Does this infinite sum have a finite value, i.e., does it converge? And if so, what is that value? A typical mathematics course will then go into sets of technical tools that allow one to prove the convergence of a series (and hopefully also reveal the value it converges to). This frequently boils down to abstract manipulation of formulae. At this point, the concept of visual proof can help. Look at the two visual proofs for convergence of the above mentioned series. From the visuals, what do you think the series converges to?

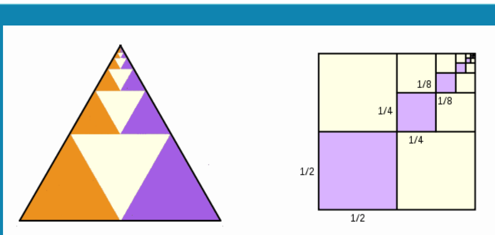

Figure 6: Two visual proofs that the sum of powers of ¼ converges. [1]

The visualizations (Figure 6) correctly reveal that the series converges to ⅓. You can verify this, for instance, in the left visualization: The triangle is split into four triangles of equal area—one is orange, one is beige, and one is purple. The last triangle is split again into one orange triangle, one beige triangle, one purple triangle, each triangle with an area of one sixteenth of the total area. The fourth triangle is split again in the same way. This continues ad infinitum. The triangles of one color represent the elements of the series. Since they divide the total area of the triangle into three equal parts, the visualization shows that this series converges to ⅓. A similar argument can be made for the other visualization on the right of Figure 6. These visual proofs provide some intuition about the convergence of the series and help us arrive at the convergence value without algebraic manipulations.

Let us turn to a second example. Figure 7 shows us how to cut a bar of chocolate, re-arrange some pieces, and finally have the entire bar of chocolate re-assembled, with one additional piece. We could repeat this process and create an infinite amount of chocolate. What else could we wish for? Well, not so fast. While creating chocolate out of thin air seems nice, it sadly violates some principles of thermodynamics. So, what is going on here, where did the visual proof of infinite chocolate go wrong?

Physically performing the trick (or some rigorous calculations) will reveal that the final re-assembled bar in the last step is slightly shorter than the bar that we started with. This shrinking provides an additional piece of chocolate. The shrinkage is minimal enough to fool us visually as the final bar seems to be of equal size. Take this example as a warning that illustrations, while helpful, can also be misleading [1].

[1] For another example of a misleading visual proof, see Joel Hamkins' essay at https://jdh.hamkins.org/all-triangles-are-isosceles/.

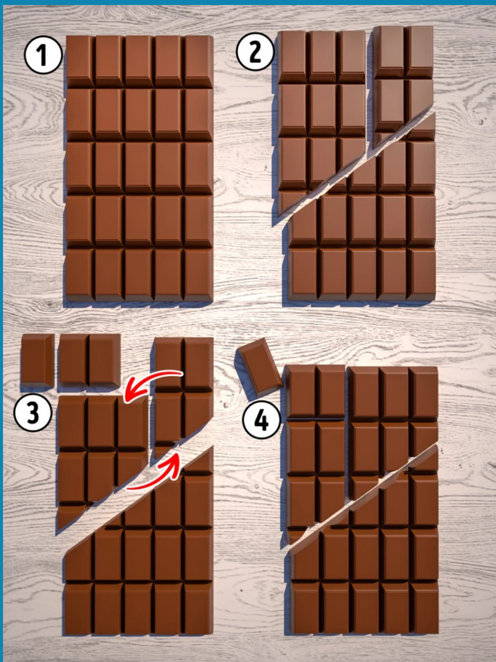

Figure 7: The "infinite chocolate" trick [2].

These examples are glimpses into the power of illustration in mathematics: they highlight how visual representations are more than just aids—they are essential tools that reshape our understanding of complex concepts. The art of math illustration transforms abstract ideas into accessible forms, fostering new insights that can even inspire breakthroughs in research[2]. Yet illustrations can mislead and have therefore to be treated as objects of research to understand both their powers and their limitations.

When we say "illustration", we use the term to encompass any of the many ways one might bring a mathematical idea into physical form or experience, including computer visualization, 3D printing, and virtual reality, among others. Today, modern technology for the first time places the production of far more complicated illustrations within the reach of many individual mathematicians. And indeed, there are a lot of illustrations to be found and explored. A prominent outlet is the "Bridges" conference for mathematics and the arts[3] or the "Journal of Mathematics and the Arts"[4]. Their respective archives are treasure troves of mathematical illustrations, see Figure 8. Aside from these, several collections of illustration projects exist, also in printed form[5].

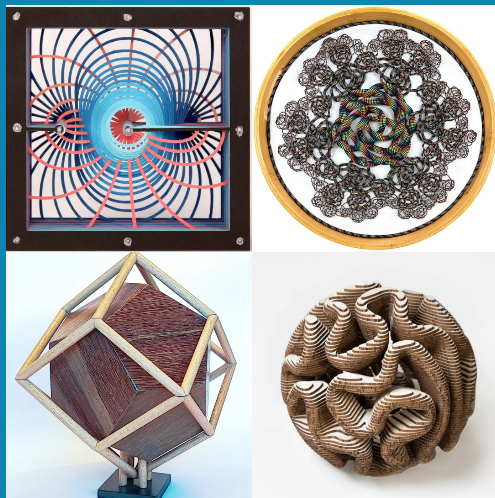

Figure 8: All taken from the "Bridges 2024 Exhibition of Mathematical Art, Craft, and Design".

Top left: **Caroline Bowen: "The Real and Imaginary Parts of ArcSech(z)"**,
Top right: **Carol Wang - "Warlow's Fractal Mat"**,
Bottom left: **Josep Rey Nadal - "Dos dotzes (Two twelve)"**,
Bottom right: **Söhnke Vetter - "Stratify"**

All these examples stand as monoliths, created from personal experiences and individual insights. They showcase the skill and creativity of their authors but are often developed in isolation, without a shared foundation for what constitutes a 'good' mathematical illustration. Despite the importance of visual representation in mathematics, there is surprisingly little overarching literature that systematically explores how to design effective mathematical illustrations. This absence of best practices leaves illustrators to rely on personal trial and error, making it challenging to learn from each other and avoid common mistakes.

[2] For more information, please read *On the importance of illustration for mathematical research* (2023) [3] and *The art of illustrating mathematics* (2022) [4].

[3] See https://www.bridgesmathart.org/ for the organization. All articles in the archive, http://archive.bridgesmathart.org/, are open access and the art galleries are also freely accessible online: http://gallery.bridgesmathart.org/.

[4] https://www.tandfonline.com/journals/tmaa20.

[5] Consider, for instance, *Mathematics and art: Mathematical visualization in art and education* (2002) [5], *Math Art: Truth, Beauty, and Equations* (2019) [6] and *Illustrating mathematics* (Vol. 135) (2020) [7].

While illustrations in mathematics may currently stand as isolated creations, we can look to the evolution of other fields to see what's possible. Diagrams, graphs, and charts began as specific tools within disciplines like physics, biology, and data science. Over time, however, they developed into comprehensive fields of study with established structures, theories, and even dedicated conferences[6]. Today, the fields of data visualization and scientific illustration have unified principles, shared methodologies, and communities of practitioners who push these disciplines forward[7]. The same trajectory is possible for mathematical illustration: with enough collective effort and shared knowledge, it, too, could evolve into a fully recognized field, complete with its own standards and a robust body of research.

Regarding research mathematics or mathematical education, illustrations quickly become scarce. Illustrations are considered as 'dangerous' as they might create false intuitions. To that, e.g., Bill Casselman writes: "Even small errors in a drawing can be confusing, frustrating, annoying, and distracting—and an accumulation of them can be deadly" [9]. This stance goes back to the Bourbaki movement in mathematics: A group that "carefully avoided using illustrations, favoring a formal presentation based only in text and formulas" [10]. Even now, these reservations towards illustrations are still present in formal mathematics.

Opposed to this, data visualization and scientific illustration provide clear examples of how a field can develop beyond initial concerns about misrepresentation or oversimplification. With a growing number of projects, these fields have gained the volume of work needed to study how users interpret visuals, assess what succeeds or fails, and refine their methods. These fields have established a nuanced understanding of how visuals impact comprehension and intuition through targeted user studies and iterative design.

Applying these principles to mathematical illustration would allow for a similar approach: by examining many projects, we could build a systematic understanding of how illustrations support or potentially mislead mathematical intuition. This knowledge would empower us to harness the strengths of illustration effectively while remaining alert to its limitations, using visuals as a powerful, intentional complement to traditional mathematical discourse.

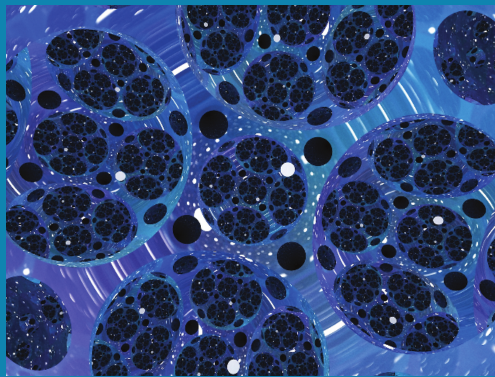

Figure 9: In-space view of a finite-volume hyperbolic 3-manifold lit by a single white light, Rémi Coulon, Sabetta Matsumoto, Henry Segerman, and Steve Trettel.

Pursuing such development is the goal of the "Illustrating Mathematics" group. Their mission statement reads: "Illustration reveals the hidden structures of mathematics, broadening access to its inherent beauty and pushing the boundaries of research. We seek to enhance professional support and recognition for illustration, pursuing mathematical depth and high-quality communication"[8].So far, there have been two community events at the Institute for Computational and Experimental Research in Mathematics (ICERM) in 2016 and 2019.

Figure 10: A fully constructed W-P foam model, Edmund Harris.

[6] https://diagrams-conference.org/

[7] Consider the IEEE Vis conference or guidelines like *Building science graphics: an illustrated guide to communicating science through diagrams and visualizations* (2022) [8].

[8] https://illustratingmath.org/

See the website of the 2016 event here and the 2019 event here. Refer to a description of the 2019 outreach activities here: Skrodzki, M. (2023). Science communication and outreach events during the illustrating mathematics semester program at the institute for computational and experimental research in mathematics (ICERM). In *Handbook of Mathematical Science Communication* (pp. 165-181). Figures 9 and 10 shows a virtual reality experience and a larger model of a space-filling foam, both created during the 2019 event.

If you want to get involved with the activities of the Illustrating Mathematics group, here are some options for you:

1. Join the "Illustrating Mathematics" Discord Server: Whether you're just starting out or are already well-versed in illustration, the "Illustrating Mathematics" Discord server is the place to connect. Here, you can ask questions, brainstorm ideas, get feedback, and share your work with a supportive community of like-minded creators and enthusiasts. This space offers an open, collaborative environment where illustrators of all levels can learn and grow together. An invite link can be found on the homepage: `https://illustratingmath.org/`.

2. Attend the Monthly Online Seminar: For those eager to dive deeper into the techniques and theory behind mathematical illustration, the monthly online seminar is a perfect opportunity. Each session explores various illustrating methods, showcases projects, and offers practical insights from experienced illustrators and mathematicians. Additionally, in a "Show-and-tell" segment, it offers the opportunity to ask questions to the community and get advice for your own illustration projects. This regular seminar is an ideal way to stay up to date on new developments, learn in-depth techniques, and engage in inspiring work in the field. Find the next seminar talks announced here: `https://illustratingmath.org/node/42`.

3. Look Out for Community Events in 2025 and 2026: The excitement continues with community events on the horizon! We're planning gatherings in summer 2025 at ICERM (Institute for Computational and Experimental Research in Mathematics) and in 2026 at the IHP (Institut Henri Poincaré). These events will offer hands-on workshops, discussions, and networking opportunities, bringing together a vibrant community dedicated to the art and science of illustrating mathematics. Mark your calendar for the 2025 event: `https://icerm.brown.edu/program/topical_workshop/tw-25-imre` and keep an eye out for a longer trimester at IHP in 2026.

References

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[10] Wikipedia contributors. (2024, August 21). *Nicolas Bourbaki*. Wikipedia. `https://en.wikipedia.org/wiki/Nicolas_Bourbaki`