Mathematical Sculptures

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Abstract: This article collection explores the rich interplay between mathematics and sculpture, showcasing how abstract mathematical ideas can take on tangible, aesthetic form. Contributors reflect on the creative processes, inspirations, and conceptual frameworks behind their mathematical sculptures, revealing diverse approaches that bridge disciplines. Together, the works highlight the power of form to communicate, question, and celebrate mathematical thinking through artistic practice.

Introduction

The fusion of mathematics and sculpture yields a captivating realm where abstract concepts take on tangible forms. Throughout history, civilizations have sought to express mathematical principles through sculpting, building, and assembling, creating fascinating works. People from ancient civilizations expressed mathematical precision in architectural marvels adorned with geometric patterns and symmetrical designs. Think of the pyramids in Egypt or Southern America, the symmetric and hierarchical construction of pagodas in Eastern Asia, or the intricate pattern in mosques throughout the Middle East. These early manifestations laid the groundwork for a tradition of mathematical sculpture that continues to inspire artists and mathematicians alike.

In the modern era, artists have continued pushing the boundaries of mathematical sculpture, by harnessing digital technologies and innovative techniques to create awe-inspiring works of art. A culmination is the https://mathemalchemy.org/ project, a brainchild of mathematician Ingrid Daubechies and fiber artist Dominique Ehrmann. It combines several art pieces into one sculpture, ranging from textile, over wooden, to clay elements. Aside from such team efforts, individual figures stand out in producing mathematical sculptures. Bathsheba Grossman is renowned for her sculptures exploring fractals and symmetry. George W. Hart’s geometric sculptures challenge perceptions of form and space. These examples highlight the creativity and ingenuity of contemporary mathematical sculptors.

Figure 1: Mathalchemy Kollektiv: Mathalchemy (2022). Photo: Kevin Allen.

Sculptures, in general, have played a vital role in public art, both as individual objects and arranged as collections in so-called sculpture parks. Many of the sculptures found in public places involve relations to mathematics. Some do so with clear intention, like the *Dodekaederstern* (German for *Dodecahedron Star*) in Vienna, Austria by mathematician Herwig Hauser. Other sculptures, such as many of artist Sol LeWitt (e.g., *HRZL #7* in Bielefeld, Germany, or *Incomplete open cubes* temporarily on display in New York City) visualize their mathematics more indirectly. The artist’s intent to use mathematics to create the sculptures (or structures, as Sol LeWitt preferred to call them) is less clear. Independent of the intent, mathematical sculptures provide opportunities to make mathematics more tangible to the broader public, exemplified in the present article collection.

(Mathematical) Sculptures can be classified by the materials used to build them, e.g., wood, metal, or stone, which surely provides an interesting perspective. We have discussed “wood” as a medium for mathematical art in a prior w/k article collection. As part of his doctorate, Ricardo Zalaya Báez proposed a different classification, based on the sculpture’s underlying mathematical content: *geometry*, *calculus*, *algebra*, *topology*, or different mathematical concepts. Each of these types can be further divided into subgroups. For instance, geometric sculptures can involve *polyhedra* (such as Sol LeWitt’s *HRZL #7*) or *fractal geometry*, while *topological* sculptures may involve *non-oriented* surfaces (such as the well-known Möbius strip) or *knots* and interwoven figures (see [1] for the complete classification). The *Dodekaederstern* would be an example of a sculpture involving different mathematical concepts, suchas *algebraic geometry*, an amalgam of algebra and geometry.

Figure 2: Herwig Hauser: *Dodekaederstern* (2013). Photo: Herwig Hauser.

The three articles in this Nudos Salvajescollection hint at the diversity of mathematical sculptures. They follow the same structure to highlight similarities and differences. The respective projects have all been presented in the minisymposium *Mathematics and Arts* at annual meetings of the German Mathematical Society.

Mathematician Aubin Arroyo and artist Jean-Michel Othonel discuss *Wild Knots* whose sculptural representations can be classified as *topological* sculptures involving *knots*. Particularly fascinating is that they studied *wild knots* independently: Aubin Arroyo from a mathematical perspective, and Jean-Michel Othonel from an artistic one. They later became aware of their respective work and started a fruitful interdisciplinary collaboration.

The next project *Polyplane* is a sculptural illustration of a formula for *polyhedra*. As such, it can be classified as a geometric (polyhedral) sculpture. The mathematicians Alex Kontorovich and Glen Whitney built *Polyplane* collaboratively with other (mathematical) artists. Specifically interesting is the use of *polyhedra* as sculptural elements that illustrate a mathematical formula about *polyhedra*; thereby, the work becomes a circular reference to its building blocks.

The sculpture *Twenty-Seven* by mathematicians Gabriel Dorfsman-Hopkins and Daniel Rostamloo is built from lines arranged in space. These are inherently *geometric* concepts, placing the sculpture in the corresponding category. However, the arrangement of the lines follows complex *algebraic* reasoning, making the final product also representative of the algebraic category. Specifically interesting is that the piece chooses to show only lines, but not the underlying surface structure in which they reside, thereby using absence to allude to the structure of the underlying mathematics.

[1] Báez, Ricardo Zalaya. A Proposal for the Classification of Mathematical Sculpture. Proceedings of Bridges 2007, https://archive.bridgesmathart.org/2007/bridges2007-67.pdf

Wild Knots

Aubin Arroyo, Jean-Michel Othoniel

Introduction

A *mathematical knot* is the representation of the abstract idea of a line that has a circular structure by connecting its beginning and end in space. It is not a perfect circle; rather, it is a curve that winds through space without ends or self-intersections. Two knots are considered *identical* if one can be manipulated into the exact shape of the other without breaking it. Despite the infinite variety of knots, mathematics provides examples that, because of their complexity, evade traditional knot theory methods: *wild knots*.

These knots are a fascinating subject in *topology* where the concept of infinity plays a crucial role, both in their characteristics and conception. However, some of them can be constructed as the limit of successive reflections in various spherical mirrors appropriately placed in space.

The mathematical aspect of the *Wild Knots* project suggests the potential to explore this process through sculptures and digital graphic materials. This is possible because the mathematical construction can be precisely translated into sculptures, enabling the viewer to mentally replicate the process and construct an approximation of these objects in their mind.

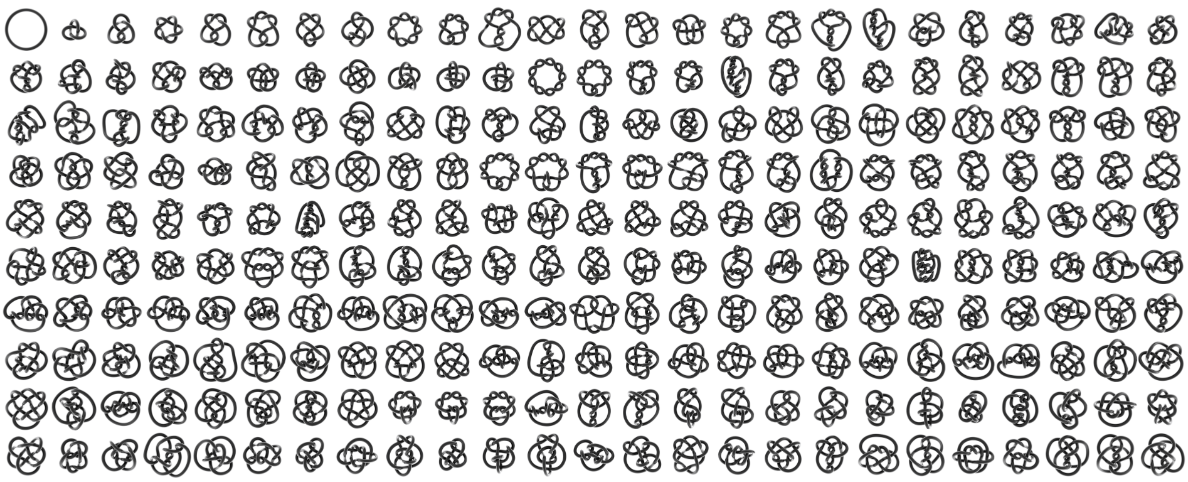

Figure 3: Aubin Arroyo: *Knot Table* (2019).

Motivation

Infinity is a perturbing idea that triggers many questions, sensations, and perplexities. Mathematics provides tools to grasp them, although they require engagement with an abstract logic-filled language. The fact that the mathematical construction can be physically realized in an artwork without the need for language, as it is implicit in the sculpture, effectively bridges mathematics and the spectator, blending both logic and emotion.

The construction of these *wild knots* is the following: The circular structure that is the basis of the knot is a necklace of shining pearls that each perfectly reflects its surroundings. The reflection of the whole necklace, in each of the spheres, describes a new necklace, also formed by spheres but smaller, and each one reproducing a copy of the original knot. This new necklace can undergo the procedure again. And this procedure repeats infinitely, amplifying its complexity with each step.

This construction is the vehicle for mathematics. Its physical representation, whether in sculpture or in graphic work or digital animation, is a recipe that, by its nature, invites the viewer to perform it in their mind. The recipe never ends. However, the result is the curve that remains within each of the collars obtained in this construction—an infinitely intricate *topological knot* termed *'wild'* due to its infinite complexity.

Figure 4: Aubin Arroyo: *Trefoil Knot: 2,352 spheres and their reflections* (2017); *Trefoil Knot: 58 spheres and their reflections* (2016); *Trefoil Knot: 870 spheres and their reflections* (2017).

Medium

The sculptural representation of this construction of a *wild knot* is faithful and requires minimal simplification in mathematical explanation to recreate the mathematical idea behind it. The concept of infinite reflections is inherently included from the moment two mirrors are placed facing each other. Computer visualization, on the other hand, allows the viewer to trace this infinite process in unimaginable detail, mirroring the mathematical construction where the necklace is reproduced within the interior of the spheres. This partially overcomes the limitation posed by physical reflection confined to the flat surface of the spheres. Digital visualization offers a significantly more detailed approximation of the infinite complexity inherent in a wild knot.

Figure 5: Jean-Michel Othoniel: *Treasure Gardens*, SEMA, Korea (2022); *Nudos Salvajes*, CCK, Argentina (2019); *My Way*, Museum of Art, Macao (2011).

Result

Introducing a mathematical idea within an exclusively artistic setting provides an alternative approachto mitigate the challenges associated with engaging in mathematics. By embedding mathematicalconcepts within an artwork, the viewer, aware of the mathematical involvement yet not immediatelyrecognizing the familiar aspects, is prompted to embrace the desire to comprehend both themathematical principles and the artwork itself. Achieving this understanding is no simple task,especially in the context of mathematics. However, on a profound level, it serves to reinforce the ideathat mathematics is a fundamental part of human culture.

Lessons

Every bridge connects two sides. The collaboration in this project arises from dialogue and the search for elements of coincidence that resonate with both sides of the bridge, centered around an abstract construction that also has two facets: one emotional and the other logical. As a result, we learn that by standing at this midpoint, one can exhibit mathematical objects with their astonishing characteristics, without the need for technical language. On the other hand, mathematical content permeates sculptures and artwork with scientific truth, giving them a new interpretative dimension [1].

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[1] Bravo, Andrea. *Estética y matemáticas: un encuentro afortunado*. Revista Capitel. No. 19. (2020), pp. 131–133.

Polyplane

Alex Kontorovich, Glen Whitney

We constantly encounter three-dimensional shapes composed of flat sides, straight edges, and pointy corners – buildings, tents, dice, prisms, or even the Great Pyramid at Giza, to name a few. Just as the everyday act of throwing a ball or dropping a coin has natural laws that it obeys, so too do these shapes have their own, mathematical, laws. In particular, one basic human ability is to count things. So a natural way to begin a search for “laws of shape” is to count the numbers of sides (represented as F, for the math term *faces*), of *edges* E, and of corners V (aka *vertices*). Looking at these numbers for many different ordinary solids (math term: *simply-connected polyhedra*), a pattern emerges. It turns out that

$F - E + V = 2$

for all of these shapes. The *Polyplane* sculpture exhibition physically embodies this *Euler Polyhedron Formula*.

Motivation

Francesco Maurolico first wrote down the *Euler Polyhedron Formula*, almost a half millennium ago. Since that time, Descartes in the 1600s, Euler in the 1700s, Gauss in the 1800s, Chern in the 1900s, Perelman in the 2000s, and countless other renowned and less-renowned researchers have made important discoveries related to or inspired by it. Rarely does one have the opportunity to present such a central and meaningful mathematical law in a way that requires no formal math experience to appreciate. Alex realized such a possibility existed here, during a meeting exploring exhibit ideas for New York's National Museum of Mathematics. He was immediately enthusiastic to spread this idea; by analogy to the musical world, being able to share Euler's insight broadly is akin to sponsoring a public performance of a Beethoven symphony.

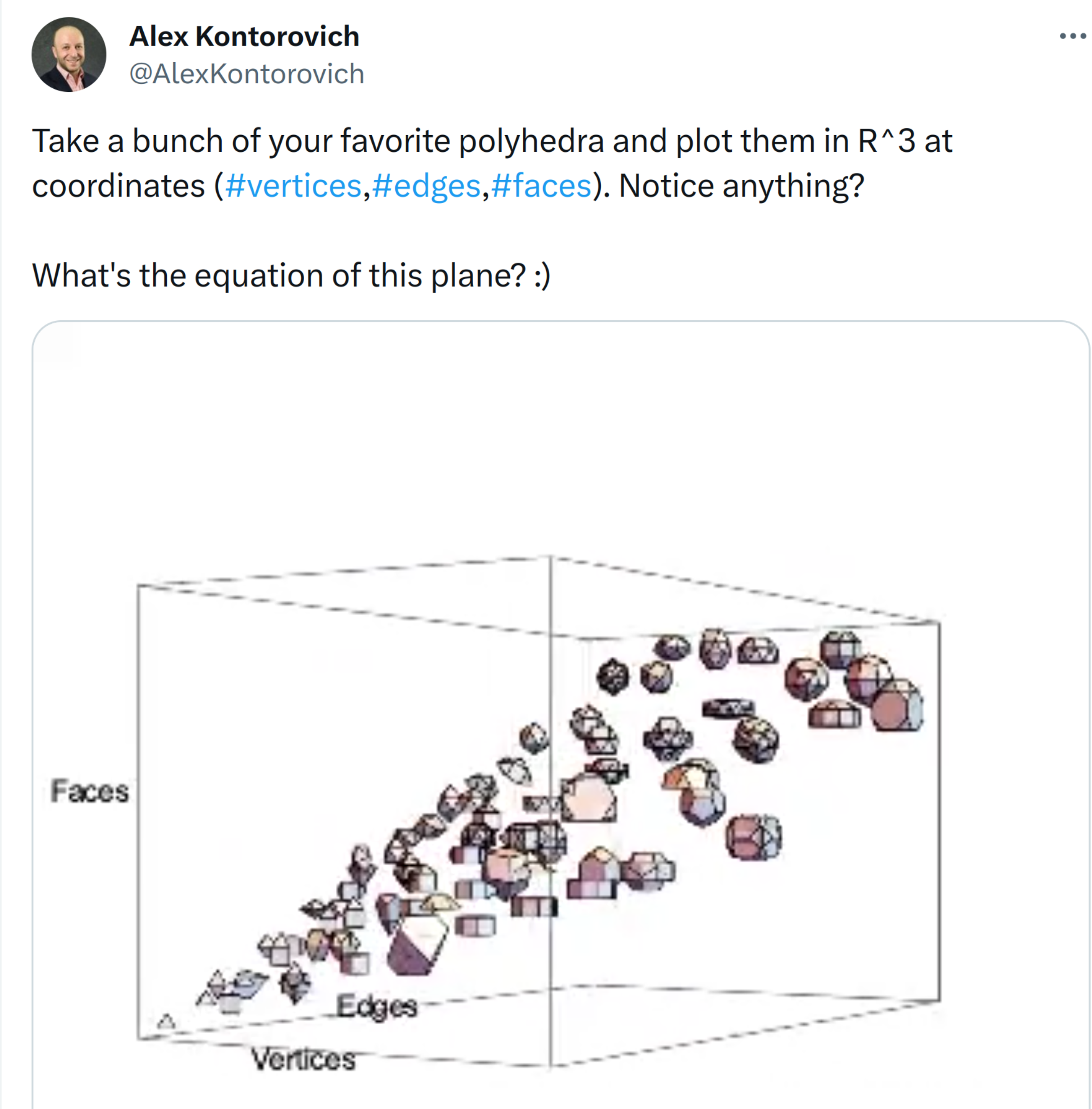

Figure 6: Alex Kontorovich: *Proto-Polyplane* (2019).

Medium

The first rendition of *Polyplane* consisted of a computer-generated graphic shared on social media (see Figure 6). Upon hearing the idea from Alex, Glen reacted immediately: It demanded to be physicallybuilt as a sculpture, or more precisely a collection of sculptures. Moreover, it needed a scale allowingvisitors to walk through the space occupied by the *polyhedra*.

Alex had an insight that makes this medium ideal. We can view any similar equation involving three quantities as the relationship among points in a plane in three-dimensional space. When each *polyhedron* is positioned horizontally corresponding to its number of faces, at a depth into the exhibit determined by its number of *edges*, and vertically by its number of *vertices*, they are compelled by the *Euler Polyhedron Formula* all to lie on a single flat plane (see Figure 7).

Figure 7: Alex Kontorovich: *Polyplane* (2023).

Outcome

The authors also wanted *Polyplane* to showcase the incredible diversity and beauty of *polyhedral* shapes, which are then elegantly unified (conceptually and physically) through the simple formula. The math and geometric art communities responded gratifyingly to this goal, with over two dozen artists contributing pieces rendered in media as diverse as carved wood, 3D printed plastic, folded paper, ceramic, stained glass, ribbon, and more (see Figure 8, or visit https://polyplane.org/gallery). Moreover, the crucial planar shape of the arrangement, carrying its central mathematical story, can easily be observed. On the practical side, although conceived and operated as a traveling exhibition, the rigors of transport have taken a steady toll, with components lost, bent, or smashed for nearly every venue to date.

Figure 8: Alex Kontorovich: *Polyplane* (2023).

Lessons

We'd like to share the two most significant points gleaned from touring *Polyplane*. First, most viewers lack a context in which the specialness of the arrangement strikes them in a visceral way. Visitors have not seen enough information plotted on three axes to recognize the miracle of alignment: without a relationship like Euler's formula, the points always come out as an unruly cloud. Second, we have gained great respect for the tension between artistic freedom on the one hand, and didactic purpose on the other. Contributors have played with the concepts of straight edges and flat faces in ways that are intriguing, expressive, and innovative, but which, for some visitors, obscure the underlying structure we are hoping will become manifest. Thus, we hope that *Polyplane* is just the beginning of presenting the laws of shape in physical form. Future companion pieces and descendants can build on these lessons and more clearly convey the central discovery while continuing to open eyes to the beauty inherent in math.

Twenty-Seven

Gabriel Dorfsman-Hopkins, Daniel Rostamloo

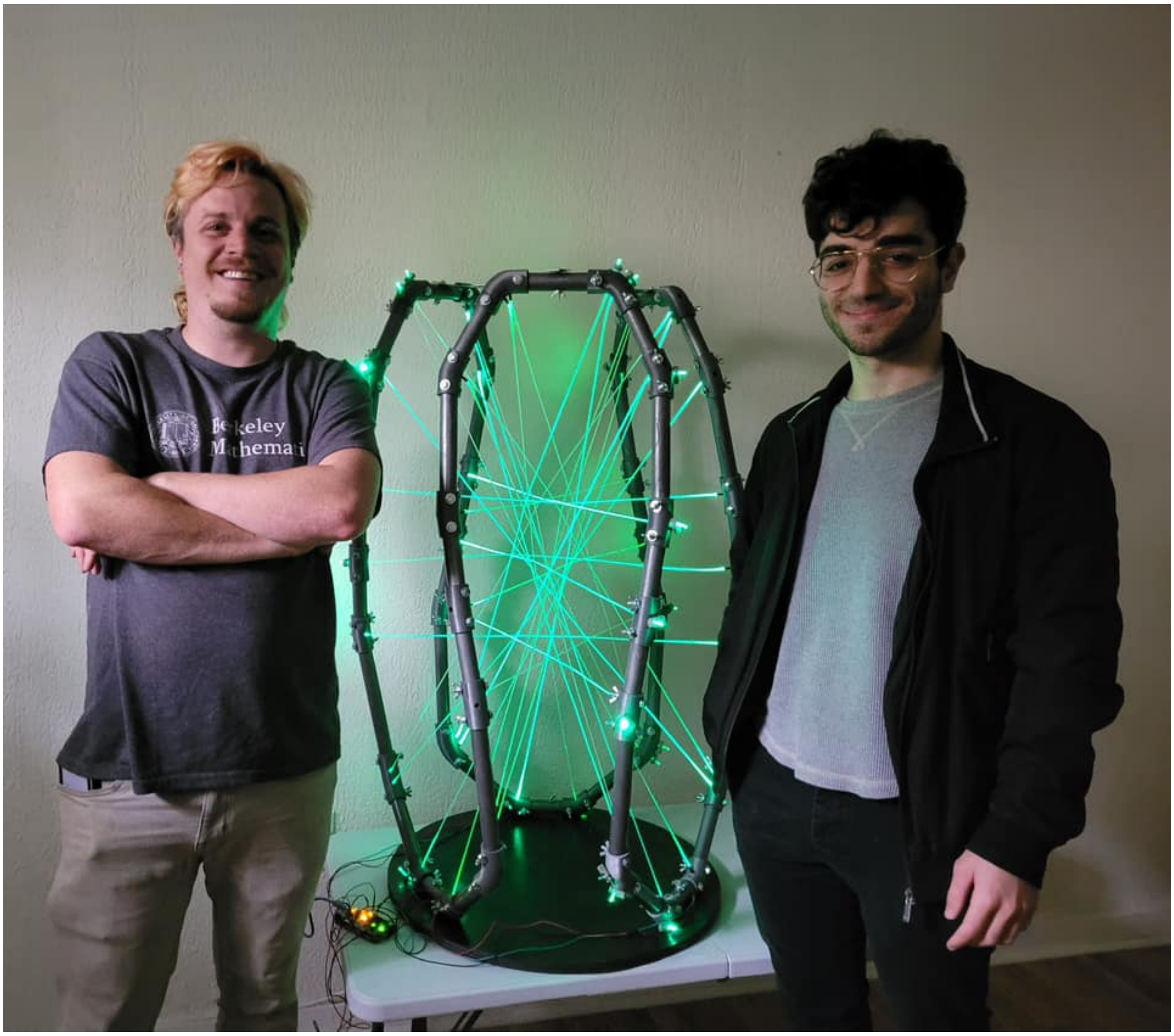

Figure 9: Gabriel Dorfsman-Hopkins and Daniel Rostamloo: *Twenty-Seven* (2023).

Introduction

A much-celebrated result of classical *algebraic geometry* is Arthur Cayley and George Salmon’s theorem which characterizes an important family of geometric objects – *cubic surfaces* – in terms of the number of lines that can be embedded into each such surface. *Cubic surfaces* are a type of *algebraic surface*, which are geometric objects obtained by plotting the solutions to an equation in the variables x, y, and z. For example, below is a picture of an *algebraic surface* (in blue) and two lines (pink and purple). Notice that the purple line is entirely contained in the surface, while the pink line only intersects it at two points.

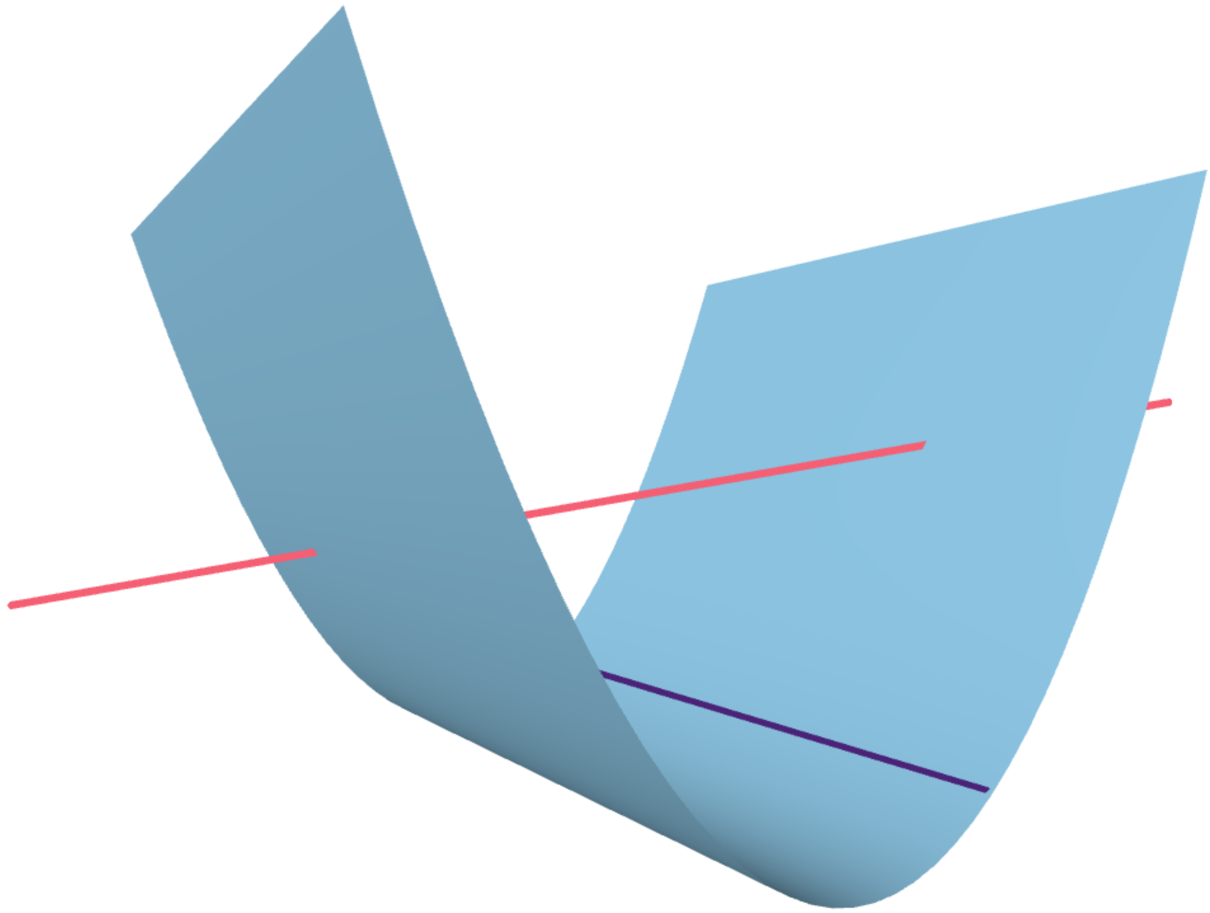

Figure 10: Gabriel Dorfsman-Hopkins and Daniel Rostamloo (2013): A degree-2 surface and 2 lines, one contained in the surface (purple) and one not (pink).

The surface in Figure 10 is given by the equation $z=x^2+2xy+y^2$. The highest exponent appearing among the terms of the equation is 2 – this is referred to as the *degree* of the algebraic surface. The surface in Figure 10 contains infinitely many lines, if we instead consider an equation in three variables with degree 3, then *Cayley and Salmon's theorem* tells us that the corresponding surface contains exactly twenty-seven lines [3] (given some necessary technical assumptions involving *complex numbers*, *projective geometry*, and *smoothness*, which are outside the scope of this article).

Motivation

This project was largely inspired by past sculptures representing this theorem through a visual and artistic medium. Early depictions date back to the 19th-century plaster models like the one in Figure 11(a). More recently, 3D printing and computational design have given new life to models and sculptures of algebraic surfaces, like those designed by Oliver Labs in Figures 11(b) and 11(c).

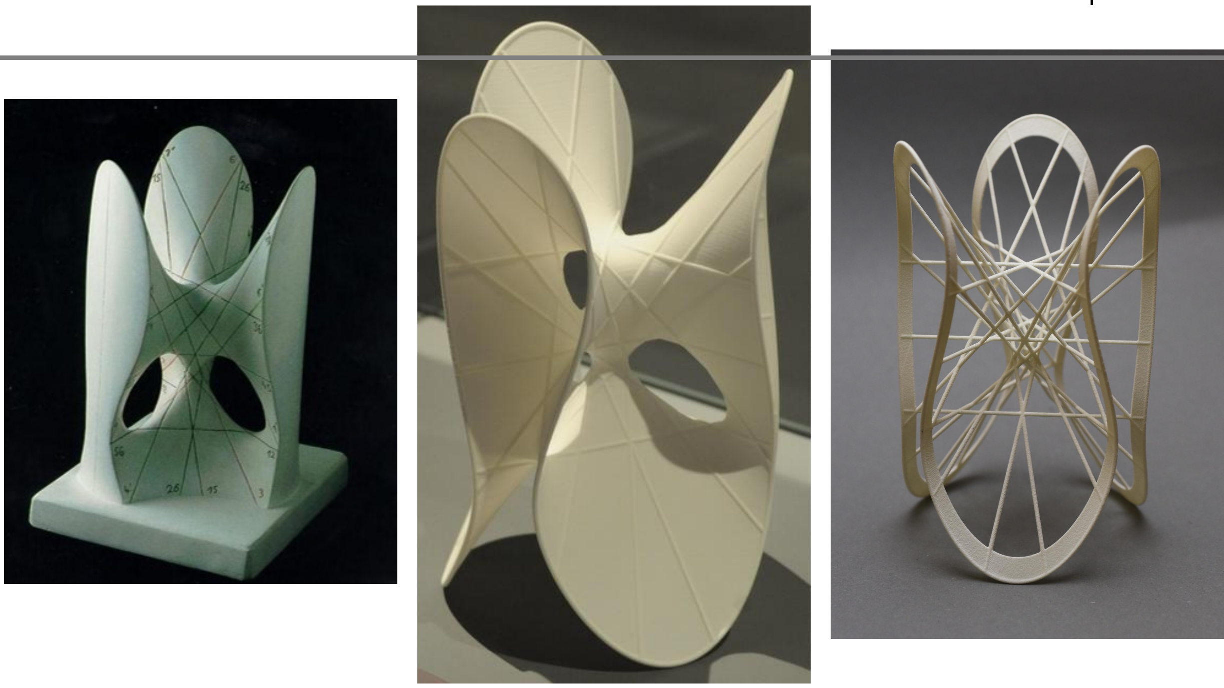

Figure 11: From left to right: (a) a plaster model of a *cubic surface*, with its 27 lines engraved, part of the *Plaster Model Collection* of Georg-August-Universität Göttingen. (b) A 3D printed *cubic surface*, with its 27 lines embossed, and (c) a 3D printed *cubic surface*, with everything except the 27 lines removed. Both 3D printed models were designed by Oliver Labs: https://oliverlabs.net/math-objects/.

We aimed to produce a larger-scale, interactive installation that builds on these previous iterations. Drawing from Oliver Labs' print (Figure 11(c)), we decided to emphasize the twenty-seven lines of the *cubic surface*, leaving the viewer to fill in the rest of the surface's interior. Second, inspired by the *algebraic surfaces* of silviana amethyst (Figure 12, [1]), we chose to use embedded electronics to augment the interactivity of the piece.

Figure 12: A 3D-printed Barth Sextic, illuminated with LEDs [1].

Medium

One of our main goals in choosing a medium was to provide an interactive viewing experience. The sculpture was originally designed on a much larger scale – around ten feet tall – and intended to be fabricated from metal rods and installed at the annual Burning Man festival (Figure 13), where participants could climb the sculpture and move through the spaces between the lines. Unfortunately, these plans could not be realized due to insufficient funding. We instead chose to scale down our design and implement interactivity by using programmable electronic lighting. This design choice accents the straight lines against the curvature of the exterior frame and allows lighting programs which depict the effects of various symmetries of the cubic surface on the lines.

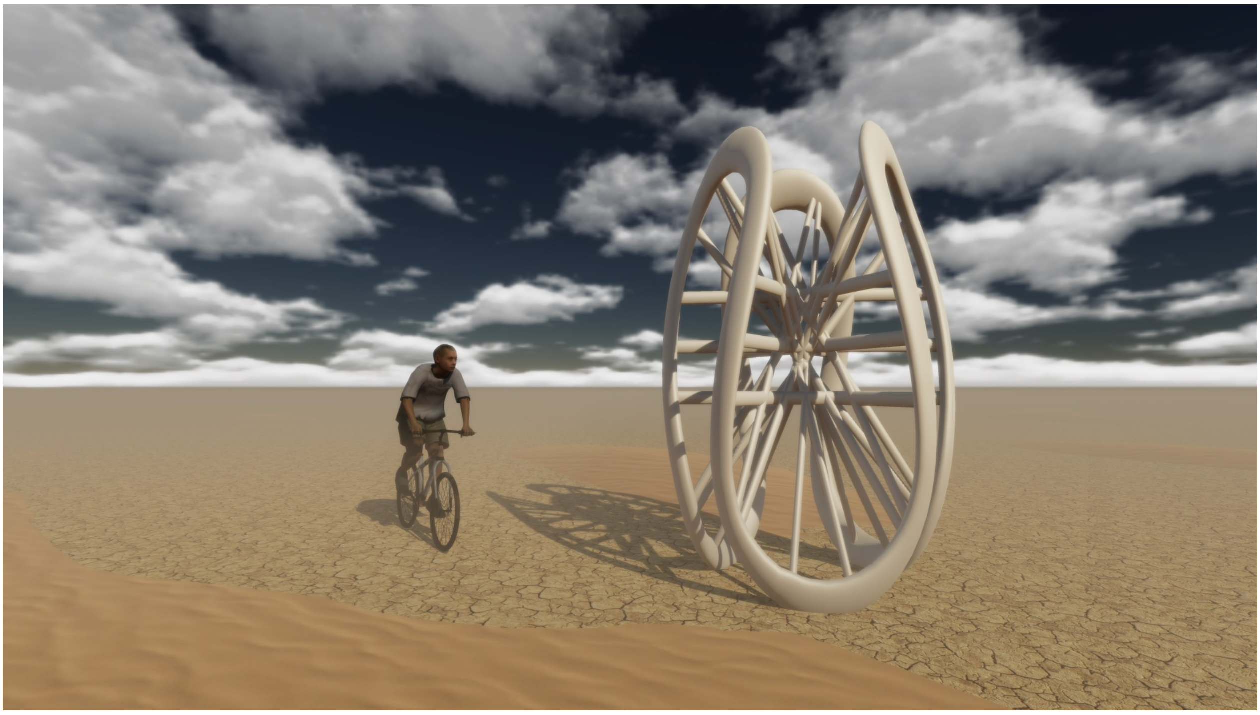

Figure 13: A render of Twenty-Seven at Burning Man

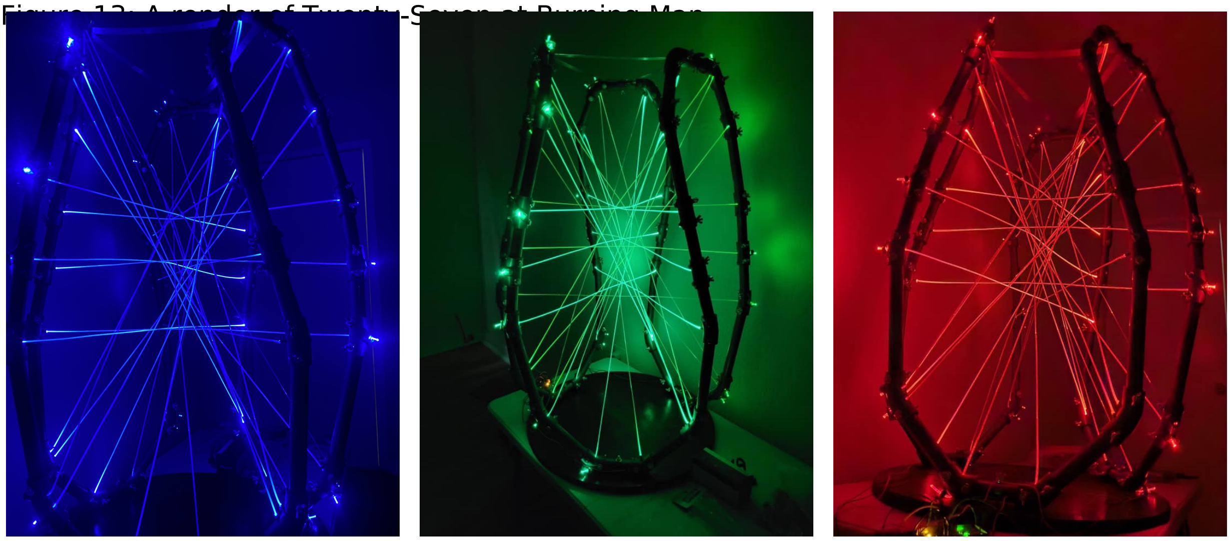

Figure 14: Three pictures of the sculpture *Twenty-Seven* by Gabriel Dorfsman-Hopkins and Daniel Rostamloo.

Outcome

In the completed sculpture, titled *Twenty-Seven*, the 27 glowing lines stand out boldly, highlighting the theorem within (Figure 14). Although the interior is empty except for the lines, our minds fill in the gaps, intuiting the curvature of the surface into which the linear skeleton is embedded. The scale and lighting create a striking visual experience, especially when the sculpture is viewed in a dark environment. Furthermore, the circuitry allows each line to be lit individually, opening the door for interactivity and the illustration of deeper and more dynamic mathematical properties. For example, we can demonstrate how the *symmetry group* of the surface acts on the 27 lines by changing their colors to track their movement.

We can also use the lighting to visualize consequences of the *rationality* of the *cubic surface*. A theorem of Clebsch [2] proves that, away from 6 of the 27 lines, a *cubic surface* is *isomorphic* to a plane. The near *isomorphism* (or, *birational mapping*) compresses these 6 lines to points and transforms 6 more lines into *conics* while preserving the *linearity* of the remaining 15. *Twenty-Seven* can use its lighting to color each line accordingly.

Lessons

This project is still in progress. One reason is technical: the most challenging aspect of building *Twenty-Seven* was incorporating the electronics and LED lighting in a large structure while maintaining the ease of transport and installation. We learned many lessons for how not to do this. We also still plan to build a larger version of this sculpture out of metal and this project is, in part, a first step toward this goal.

There are also more mathematical avenues to explore. While searching for a model for our sculpture, we realized that we could create families of models based on selections of points in the plane (using a process called *blowing up*). This inspired us to design families of these sculptures, each a slight deformation of the last, using the lighted lines to keep track of the parameter we modify, and we plan to create a family of cubic surfaces within this same design paradigm.

[1] amethyst, silviana; Maurer, Samantha; O'Brien, William. A 3D printed Arduino-powered interactive Barth Sextic. 3D printing in mathematics, 51–72. Proc. Sympos. Appl. Math.,79.

[2] Clebsch, Alfred and Gordan, Paul. Theorie der Abelschen Functionen. Leipzig B.G. Teubner, 1866.

[3] Salmon, George. A treatise on the analytic geometry of three dimensions. Vol. II., Fifth edition, Longmans and Green, London, 1915.

Tags

1. Art-Related Science
2. Martin Skrodzki
3. mathematical art
4. mathematics
5. mathematics and art
6. Milena Damrau