22 A Special Game

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Challenge

The games department of the Christmas Research Center is constantly developing new games to keep the children of the world from getting bored. This year, it has come up with a game especially for only children: it is a solitaire game played on a board with $n \times 3$ fields.

Each field can hold exactly one token. There are three different types of tokens and n tokens of each type: squares, circles, and diamonds. For each type, the n tokens are labeled with numbers 1 to n.

A placement of all tokens on the board is called *stable* if:

(a) In no row a square is to the right of a circle or a diamond.
In no row a circle is to the right of a diamond.

(b) In columns, the vertical relations on labels as given in Figure 23 hold:

That is, the numbered squares, circles, and diamonds have to be placed according to the inequalities given in Figure 23. For instance,

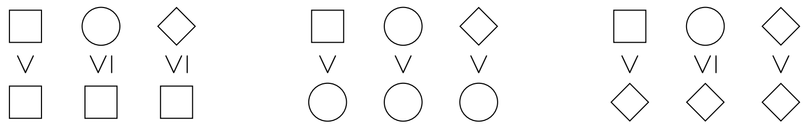


Figure 23: The vertical relations on labels.

- a square numbered i can be below a circle or diamond numbered j with $i \leq j$,
- but a if a circle numbered i is under a square, then the square's label needs to be j with $i < j$.

(c) In rows, the horizontal relations on labels as given in Figure 24 hold, even if the respective tokens are *not* next to each other:

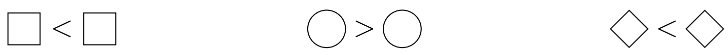
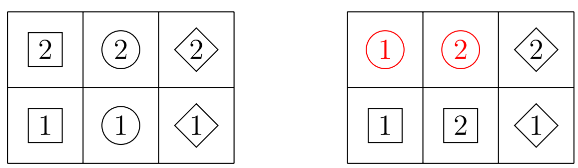


Figure 24: The horizontal relations on labels.

Examples of a stable and an unstable placement for the 2×3 board is depicted in Figure 25:



stable unstable

Figure 25: A stable and an unstable placement on the 2×3 board. The left placement of the tokens is stable; all relations are satisfied. The right placement of the tokens is unstable: while all columns as well as the lower row satisfy all relations, the upper row violates rule (c), since the circle with larger number is placed to the right of the circle with the smaller number (marked in red).

Note that while there are restrictions on labels across different types of tokens when placed *vertically*, there are only restrictions on *horizontal* labels within one token type. That is, if two tokens of the same type are in the same row, they have to be ordered from left to right according to their label (even if they are not next to each other). In rows, two tokens of different types only have to satisfy rule (a), independent of their labeling. That is, a square labeled 5 can be left of a circle labeled 3.

Question: How many *stable* placements of the tokens are there on a 4×3 board?

Possible answers:

1. 1
2. 2
3. 3
4. 4
5. $3 \cdot 4$
6. $3! \cdot 4$
7. $4! \cdot 3$
8. $3! \cdot 4!$
9. $(3 \cdot 4)!/3!^4$
10. $(3 \cdot 4)!$

Project reference:

The presented task is part of an open question in a current combinatorics project. In the project, we investigate an object called the *neighborhood grid*, which has connections to finding relationships in data sets (for more details, see: `https://arxiv.org/abs/1710.03435`). Roughly speaking, the data structure will speed-up the performance of specific algorithms such as particle or cell simulations by providing fast, approximate answers to the question: For a given particle or cell, what are the nearest other particles or cells around it?

Solution

The correct answer is: 1.

There is only one stable placement of the tokens. Checking for this solution amounts to going through all combinatorial possibilities of placing the tokens and verifying that only one solution is valid, namely that of placing all squares in the left column, all circles in the center column, and all diamonds in the right column, sorted by their labels from bottom to top. Next to brute-forcing the solutions, e. g., with the help of a computer, it is helpful to deduce limited positions for certain tokens.

Consider the square with label 1. By rule 1, it has to be left of any circle or diamond; i. e., it cannot be in a row such that it is in the second or third column with circles or diamonds next to it. Furthermore, squares in a row have to be ordered from left to right according to their label. Thereby, the square has to be left of all other squares, as it has the lowest available label.
A similar reasoning holds for the vertical placement. Squares, circles, and diamonds have to satisfy a strict inequality when being placed below a square. But the considered square has the lowest label available. Hence, it has to go in the lowest column. Combining these two observations pins the square in the lower left corner.

Now consider the diamond with label 4. By a similar reasoning as above, since the diamond has the highest label available, no diamond can be right to it. Furthermore, following rules 1 and 2, no squares or circles can be right to it. Hence, the diamond with label 4 has to be in the right column.
Furthermore, following the vertical rules from Figure **??**, a diamond has to satisfy strict inequality to squares and other diamonds. Since the diamond with label 4 has the highest label available, it can either go to the top row or sit below a circle of label 4.

One can now continue with this sort of reasoning, e. g., by placing the circle of label 4, in case it has not been fixed by the choice of placing the diamond of label 4. This provides a growing decision tree as to where the remaining tokens can go on the board.

Apart from checking all combinations of other placements, no solution that holds for general n is known yet (see project connection). Going through all possibilities reveals that a continuation of the stable state shown in Figure **??**, i. e., placing all squares in the left, all circles in the center, and all diamonds in the right column, ordered by their label is the only stable placement possible.

Relation to current research

The combinatorial question we are currently investigating is, whether for any $n \times m$ board, there is a certain class of objects such that they allow only one stable placement.

The presented exercise is a special case for $m = 4$ of the open question, which asks: For any n, is there only one stable placement for the tokens? So far, we were able to check that the answer is yes for up to $n = 7$, which amounts to checking $21! \approx 5.11 \cdot 10^{19}$ different placements. We conjecture that the presented tokens have only the one placement solution for any n; i. e., placing all squares in the left, all circles in the center, and all diamonds in the right column. However, we cannot prove it at this point.

If you have suggestions or prove ideas for how to tackle the problem, please get in touch:
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